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The elusive straight line: a structural engineering example

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ABSTRACT: In this article, the author relates and discusses recent experiences, together with the lessons that were learned from the upper-division undergraduate/Masters-level-graduate advanced structural analysis course offered at the University of North Carolina at Charlotte, in Charlotte, USA. A background of the course is first delineated and then followed by a pair of examples that illustrate typical structural analysis problems considered during the semester and the techniques involved in order to obtain solutions. Finally, a related examination question containing one fundamental difference from that of the other problems considered before – namely, that of a linearly-varying load distribution – is recounted relative to class-performance outcomes, which indicate that today's engineering students may have serious deficiencies that educators would be remiss not to take notice of, and, in earnest fashion, attempt to rectify and reform.

INTRODUCTION

Drafting effective yet equitable examinations for classes is, in many ways, an art, the brushstrokes of which are not easily conveyable. Notwithstanding this, one overarching objective to strive for as the instructor in this regard should be an attempt to successfully strike a balance between the targeting of major topics covered in the class, while managing to gauge students' command over the more minute, yet altogether pertinent, details. It is with such a setting that the author relates some recent post-examination experiences from the *Advanced Structural Analysis* class (hereafter referred to as *Advanced Structures*; formal course-listing: CEGR 4224/5224), which he was assigned to teach for the first time in the spring 2005 semester, ie his second spring semester as a new faculty at the University of North Carolina at Charlotte, in Charlotte, USA, in which he discovered that one of the most fundamental concepts so common and prevalent to all the various disciplines of science, mathematics, and engineering – that of the *straight line* – became a major stumbling-block for his entire class of 18 engineering students (breakdown: 16 BS-level seniors and two MS-level graduates).

BACKGROUND OF ADVANCED STRUCTURES COURSE

As might be inferred by the name *Advanced* in the course title, this class builds on the knowledge acquired in the more rudimentary class on structural analysis wherein students are primarily involved in the study of beams, trusses and frames that are said to be *statically determinate*, ie equations of equilibrium (most commonly written as: $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M_z = 0$, in a conventional 2D, *xy*-rectangular coordinate system [1]) are sufficient to determine all needed support reactions and internal forces in the aforementioned structures. The predominant context of the Advanced Structures course, on the other hand, is concerned with the analysis of *statically indeterminate* structures, ie those containing more support reactions and/or members than what is required for statical stability – these excess reactions and/or members are known formally as *redundants*, and the total number of these is termed the *degree of statical indeterminacy*, denoted commonly as *i* (one of the advantages for choosing to design structures in this way is to enable the redistribution of loads in the system should certain portions become overstressed or collapsed as a result of earthquakes, tornadoes, impact, etc). Consequently, the foregoing equilibrium equations alone are insufficient for purposes of analyses and must, therefore, be supplemented by additional relationships, known as *compatibility conditions*, based on the deformed geometry of the structure.

The techniques used to analyse statically indeterminate structures can broadly be divided into two categories: force (or flexibility) methods and displacement (or stiffness) methods. The latter approach involves the writing of force-displacement relations that, when considered with the equilibrium requirements, enables the solution of unknown displacements and forces; slope-deflection and moment-distribution are but two of the stiffness techniques that are commonly taught in this classification. The former method consists of satisfying compatibility and force-displacement requirements from which solutions of redundant forces and/or members are obtained.

In the Advanced Structures course, the author introduced a general formulation of the flexibility approach known as the *Method of Consistent Deformations*, which not only constituted the basis of one of the two test questions recently posed to his students on an examination – out of which, incidentally, the essence for this article was spawned – but, likewise, a *bona fide* target of one of the *major topics* he had covered in considerable depth throughout the semester, as well. Originally developed by James Clerk Maxwell (1831-1879) in 1864, it was later refined by Otto Mohr (1835-1918) and Heinrich Müller-Breslau (1851-1925) [2]. This approach ingeniously exploits the powerful method of superposition to arrive at a solution by the following procedure:

1. An adequate number of redundants (based on $i \equiv$ degree of statical indeterminacy) are removed from the original indeterminate configuration to render a system that is both statically-stable and solvable – known technically as the *primary structure*;
2. A total of $i + 1$ primary structures are then considered – one for the primary structure as subjected to the external loading only, together with a series of i additional primary structures corresponding distinctly to each of the various redundants that are now imposed upon these members as unknown loads (technically, these loads are expressed in terms of a product of a known-load of unit-magnitude and the, as yet to be determined, unknown redundant);
3. The primary structure with the external loading is then analysed for the i number of deformations, eg deflections and rotations, as consistent with each of the released redundants, while each of the i remaining primary structures are analysed for deformations (technically, flexibility coefficients or compliance values, since, by the inverse of Hooke's Law: deformation = compliance × load, where the load is taken as unity at this stage, and so deformation ≡ compliance);
4. The unknown redundants are then finally determined by solving a system of i linear equations arising from *conditions of compatibility*, ie requirements ensuring that the displacements of the primary structure due to the combined effect of the redundants and the given external loading, conform to the deformations of the original indeterminate structure.

Two examples are offered next to illustrate the aforementioned methodology involving statically indeterminate beams – both of which were solved by students as homework assignments. The first has reference to a single degree of indeterminate (SDOI) structure, and the second has a multiple degree of indeterminate (MDOI) configuration. Following these, the author then describes and discusses the specific examination problem that he recently posed to his students.

EXAMPLE 1: AN SDOI STRUCTURE

Consider the following propped-cantilever beam of Figure 1 as subjected to a uniformly distributed load, w, over its span-length, L. In this case, the total number of constraints equals 4, one horizontal, and one vertical reaction, along with a moment at the fixed-end A, denoted as A_x, A_y, and M_A, respectively, and a single vertical reaction, B_y, due to the roller support at the opposite end, B.

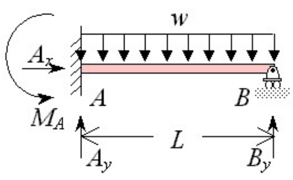

Figure 1: Propped-cantilever beam.

Since there are four reactions (ie $r = 4$) in this problem, and, as previously mentioned, the number of available equations due to equilibrium total 3 (ie $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M_z = 0$), the problem contains a degree of indeterminacy, i, of $r - 3 = 1$; the structure is hence classified as being an SDOI beam that possesses a sole redundant reaction. To analyse this indeterminate beam, the four-step methodology as outlined in the preceding section is described below:

1. Due to the presence of a *single* redundant, *one* of the reactions is selected for removal, such that the resulting *primary structure* remains statically stable and solvable; take, for instance, B_y to be the redundant in this example.
2. Next, a total of two, ie $i + 1 = (1) + 1$, primary (and statically *determinate*) structures are analysed for deflections Δ_{B0} and f_{BB} by standard means (ie double-integration, conjugate-beam, virtual-work, etc) – one, containing only the external load, w, and the other, having an imposed unit-load that is scaled-up by the, as yet to be determined, B_y redundant, to counter the displacement (see Figures 2 and 3; here, the subscript, 0, in the case of the former symbol is used to signify its relation to the primary structure having the external load, as, likewise, the second subscript, B, of the latter, ie flexibility coefficient, indicates its association with the primary structure due to the application of the redundant-force, B_y; the first subscripts in each case denote the location along the member wherein expressions for the deflections are being sought).

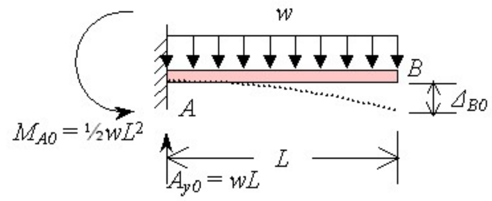

Figure 2: Primary structure with an external load, w.

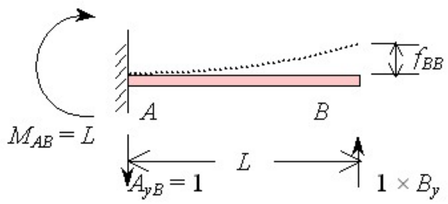

Figure 3: Primary structure with an imposed unit-load.

3. The double-integration method:

$$EI\, y'' = M \tag{1}$$

where E = modulus of elasticity, I = moment of inertia, y = deflection, and M = moment, can be conveniently used to ascertain the required deflections of both primary structures (it should be understood that other methods like virtual work, or conjugate-beam, etc, can also be used for finding the deflections). To this end, the governing moment-expressions must be determined at any point, x, along each of the beams (ie for $0 \le x \le L$) – resulting in, respectively:

$$M_0 = -\frac{wx^2}{2} + wLx - \frac{wL^2}{2} \tag{2}$$

$$M_1 = L - x \tag{3}$$

where the subscripts 0 and 1 are used to distinguish between the externally- and redundant-loaded primary structures, respectively.

To see how the foregoing moments are found, consider, as an example, the free body diagram (FBD) of the externally-loaded primary structure that is sliced at some arbitrary distance, x, away from the left support, A (see

Figure 4). In keeping with standard beam sign-conventions (eg Ref. [3]), a sagging moment is taken as being positive, ie one that has concave-upward, or positive, curvature, and, as such, holds water; also, a shearing force is positive when it causes clockwise rotation about the midpoint of the FBD upon which it acts.

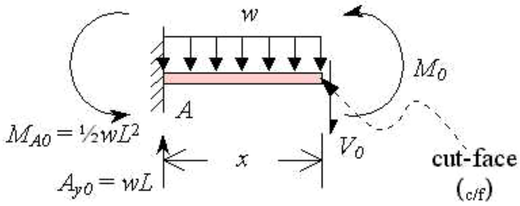

Figure 4: Primary structure with external load, w, and sliced at an arbitrary distance, x, from end A.

Hence, the counterclockwise-positive (ie CCW+) summation of moments about the cut-face (c/f), gives: $\Sigma M_{c/f} = 0$:

$$M_0 + wx(½x) - wL(x) + (½wL^2) = 0 \tag{4}$$

This can be solved for M0 to yield equation (2). Equations (2) and (3) are next substituted into (1) and integrated twice to determine the expressions for the unknown deflections, as follows:

$$\begin{aligned} EI\ y_0'' &= -\frac{wx^2}{2} + wLx - \frac{wL^2}{2}, \\ EI\ y_0' &= -\frac{wx^3}{6} + \frac{wLx^2}{2} - \frac{wL^2x}{2} + k_1, \\ EI\ y_0 &= -\frac{wx^4}{24} + \frac{wLx^3}{6} - \frac{wL^2x^2}{4} + k_1x + k_2 \end{aligned} \tag{5a, b, c}$$

$$\begin{aligned} EI\ y_1'' &= L - x, \\ EI\ y_1' &= Lx - \frac{x^2}{2} + c_1, \\ EI\ y_1 &= \frac{Lx^2}{2} - \frac{x^3}{6} + c_1x + c_2 \end{aligned} \tag{6a, b, c}$$

It should be noted that the constants of integration: k1, k2, c1, and c2, are all zero due to the boundary conditions for this problem, namely that at the fixed-end, the beam can neither rotate nor deflect, ie:

$$\text{at } x = 0,\ y' = 0 \tag{7}$$

$$\text{at } x = 0,\ y = 0 \tag{8}$$

Hence, the required deflections at the free-end of each primary cantilever can now be solved by substituting L in the place of x in (5c) and (6c), resulting in, respectively:

$$\Delta_{B0} = -\frac{wL^4}{8EI} \tag{9}$$

$$f_{BB} = \frac{L^3}{3EI} \tag{10}$$

4. The value of the unknown redundant, B_y, can now be determined through the compatibility condition that requires the deflection of the original propped-cantilever beam at the roller-support to be zero, ie $\Delta_B = 0$; but, since $\Delta_B = \Delta_{B0} + f_{BB}\ B_y = 0$, we have that:

$$\Delta_{B0} + f_{BB}\ B_y = 0 \Rightarrow B_y = -\frac{\Delta_{B0}}{f_{BB}} \tag{11}$$

Thus, substitution of (9) and (10) into (11) gives:

$$B_y = \frac{3wL}{8} \uparrow \tag{12}$$

Having now found the value of the redundant for this SDOI beam, all of the other reactions can then be found by the equations of equilibrium.

EXAMPLE 2: A MDOI STRUCTURE

Taking again the same basic problem that was just discussed, except that now, instead of providing a roller-support at end *B*, both ends are completely fixed. At first glance, it appears that $i = 3$, a pair of unknown moment, vertical, and horizontal reactions on either end, totalling six in all, versus the three equations of equilibrium; however, since the beam is subjected to vertical loading only, the horizontal reactions, A_x and B_x, must both necessarily be zero. Thus, four unknowns are now compared against two available equations of equilibrium (not three, since $\Sigma F_x = 0$ is no longer relevant), and so $i = 2$ for this problem.

1. Take B_y and M_B as the redundants.
2. The following three, ie $i + 1 = (2) + 1$, primary structures for deflections are analysed:

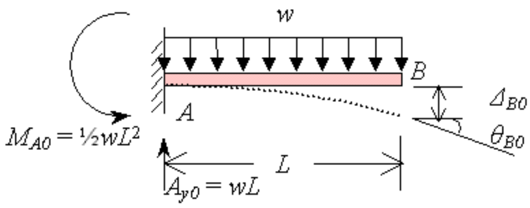

Figure 5: Primary structure with external load, w.

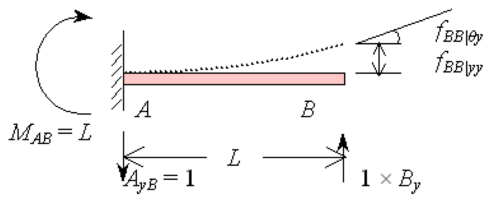

Figure 6: Primary structure with imposed unit-load.

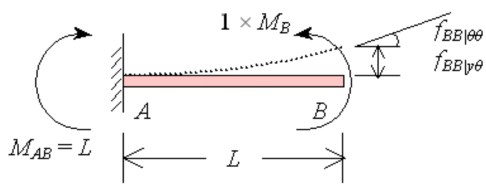

Figure 7: Primary structure with imposed unit-moment.

where $f_{BB|yy}$ and $f_{BB|\theta y}$ represent the compliance at end *B* due to redundant, B_y, that result in, respectively, a deflection and rotation arising from the imposed unit-vertical load; similarly, $f_{BB|y\theta}$ and $f_{BB|\theta\theta}$ represent the compliance at end *B* due to the redundant, M_B, resulting in, respectively, a deflection and rotation as arising from the imposed unit-moment.

3. Again, by use of the double-integration method, the following deflections can be found:

$$\Delta_{B0} = -\frac{wL^4}{8EI};\ \theta_{B0} = -\frac{wL^3}{6EI} \tag{13a; b}$$

$$f_{BB|yy} = \frac{L^3}{3EI};\ f_{BB|\theta y} = \frac{L^2}{2EI} \tag{14a; b}$$

$$f_{BB|y\theta} = \frac{L^2}{2EI};\ f_{BB|\theta\theta} = \frac{L}{EI} \tag{15a; b}$$

Note that $f_{BB|\theta y} \equiv f_{BB|y\theta}$, as expected, due to the Maxwell-Betti reciprocal theorem [4].

4. The unknown redundants, B_y, and M_B, can now be found through the compatibility conditions, which require that $\Delta_B = 0$ and $\theta_B = 0$. Hence, a pair of equations must be solved simultaneously that may be represented in matrix form, as follows:

$$\begin{bmatrix} f_{BB|yy} & f_{BB|y\theta} \\ f_{BB|\theta y} & f_{BB|\theta\theta} \end{bmatrix} \begin{Bmatrix} B_y \\ M_B \end{Bmatrix} = -\begin{Bmatrix} \Delta_{B0} \\ \theta_{B0} \end{Bmatrix} \tag{16}$$

The substitution of (13) - (15) into (16) yields, therefore, the following solutions for the redundants:

$$B_y = \frac{wL}{2} \uparrow; \quad M_B = \frac{wL^2}{12} \circlearrowright \tag{17a; b}$$

EXAMINATION PROBLEM WITH THE STRAIGHT LINE

During the class-meeting preceding the examination, the author announced his intentions to assign an MDOI structure to the students – specifically, one with $i = 2$ – which would be furnished with some of the (five) unknown deflections. In so doing, students could test their abilities to determine deflections (a most significant part of the course) while gauging their proficiency in solving a matrix-equation of the form of (16), in the limited time allotted for the examination period. Further, the author wanted to draft an examination question similar enough to the homework problem that they had already solved (see the preceding section), yet sufficiently different to evaluate their critical thinking skills. Based on these criteria, he came up with the problem shown in Figure 8, having a linearly varying load of intensity w starting at end A and tapering to zero at mid-span.

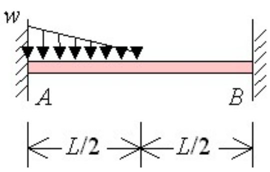

Figure 8: Fixed-fixed beam for the examination.

Prior to finalising the examination problem, the author considered furnishing his students with the load-equation; however, upon further deliberation on this possible option, he decided that engineering students (seniors and graduates, no less!) ought to be able to apply their knowledge of the line-equation to establish the correct expression that characterises the intensity of the load as a function of the distance, x, along the member. Much to his amazement, the vast majority of those in his class stumbled upon this most elementary segment of the test-question – the piece that arguably has little, if anything at all, to do with the course-content – and yet, due to a lack of appreciation and understanding of such a core-concept (commonly taught at the secondary-level), there could be no hope of them solving the remainder of the actual structural engineering problem-at-hand. Out of 18 people in the class, not one single person was able to correctly solve the problem. Even during the test-period, before the time had expired, it became quite apparent that the majority of students were not too happy. Many vocalised their despondency by complaining that there was not enough time (1.5 hours) and that the test was too long (even though there were only two questions, with the second being especially designed as a straightforward slope-deflection problem – a freebie of sorts). Upon leaving the classroom, one student even branded the test as being *impossible*. In order to dismiss such objections of inadequate time-allotment or the possibility that an anomaly as this resulted because of the students' unfamiliarity with such a problem, the author tried a small experiment. Without revealing his plan to his students beforehand, the very next class-meeting (ie two days later) he gave everyone a re-take of the mid-term – a (golden) opportunity for people to redeem themselves after having had ample time to think through the problem. One other aspect of the re-take worth mentioning is that the slope-deflection problem (ie the freebie) was eliminated so as to allow students the entire 1.5 hours to focus on the single (*impossible*) problem. With even more disappointment this second time around than the first, again, no one was able to correctly solve the problem.

Interestingly, later that same day, following the initial examination, one student told the author that they had been attempting a solution of the problem for the past three hours but to no avail. When asked to show their work, the very first thing that drew the author's attention on their scratch-paper was the following expression that they had derived for the loading-profile: $-2x + w$, the author then proceeded to explain that the expression was incorrect, since, for example, at $x = L/2$, the load based on their equation would yield a value of $w - L$, which apart from being erroneous (ie it is 0 there), is also meaningless from a units-standpoint (w is given in terms of load per unit distance, and L, in distance alone). When the author explained how the correct expression (ie $(-2w/L)x + w$ (valid for $0 \leq x \leq L/2$)) for the linearly-varying load could be obtained through the standard equation of a line ($y = mx + b$), the student commented that had they been given this information during the examination, they would have been able to successfully solve the problem.

CONCLUSIONS

Empirically it may be intimated that most undergraduate, even many graduate, engineering students, lack a general mastery over the fundamental concepts of mathematics that the instructor oftentimes assumes as existing to the contrary. It has been shown in this article how one of the most elementary and pervasive concepts within the sciences – that of the straight line – was recently the chief stumbling source for a class consisting of 18 upper-division undergraduate and Master's-level graduate structural engineering students. In conclusion, then, if one intends to responsibly educate the next generation of engineers, one must seriously consider returning *back to the basics*. More emphasis must be placed on mathematics in a curriculum – the very language underlying the engineering discipline (despite its unpopularity with students, and the author ventures, some colleagues) – if we are to graduate engineers who can successfully use these basic ideas to solve practical problems, then what remains elusive to them now, may regress into an ill-fated future for society as a whole, tomorrow.

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